On boundary conditions in three-dimensional AdS gravity

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Abstract

A finite action principle for three-dimensional gravity with negative cosmological constant, based on a boundary condition for the asymptotic extrinsic curvature, is considered. The bulk action appears naturally supplemented by a boundary term that is one half the Gibbons-Hawking term, that makes the Euclidean action and the Noether charges finite without additional Dirichlet counterterms. The consistency of this boundary condition with the Dirichlet problem in AdS gravity and the Chern-Simons formulation in three dimensions, and its suitability for the higher odd-dimensional case, are also discussed.

1 Introduction

Three-dimensional gravity with negative cosmological constant [1] is a simple model that catches the main features present in $D > 3$ dimensions. In fact, this theory –first considered by Deser and Jackiw in [1]– has black hole solutions, possesses a rich asymptotic dynamics and, as in the higher-dimensional case, its action also needs to be regularized in order to give rise to finite conserved charges and Euclidean action.

The dynamics at the boundary is determined by the asymptotic behavior of the gravity fields. Supplementing the action with appropriate boundary terms and demanding boundary conditions, the asymptotic dynamics of three-dimensional AdS gravity is described by a Liouville theory [2].

The boundary dynamics is essential for a well-posed definition of the global charges. For example, the algebra of asymptotically locally AdS gravity in three dimensions is infinite-dimensional conformal algebra described by Virasoro generators, whose Hamiltonian realization in terms of conserved charges introduces a non-trivial classical central charge [3].

Many interesting properties of this gravity theory are due to the fact that the AdS gravity can be formulated as Chern-Simons theory for $SO(2,2)$ group [4] (see also [5]). In this context, the global charges in Chern-Simons AdS gravity in Hamiltonian formalism were studied in [6].

In the framework of AdS/CFT correspondence [7, 8], the duality between AdS gravity and a Conformal Field Theory on the boundary is realized by the identification between the gravitational quasilocal stress tensor and the conformal energy-momentum tensor. In that way, the

stress tensor in the CFT generating functional couples to the boundary metric (initial data for the Einstein equation), from where the n-point functions are computed. In the AdS gravity side, this information is encoded in the finite part of the stress tensor, that needs to be regularized by a procedure that respects general covariance on the boundary (holographic renormalization) [9]. This method provides an algorithm to construct the (Dirichlet) counterterms to achieve finite conserved quantities and Euclidean action (see, e.g., [10, 11, 12, 13]).

In practice, however, this regularization procedure is easy to carry out only for low enough dimensions, because the number of possible counterterms increase drastically with the dimension. Moreover, these terms do not seem to obey any particular pattern and the full series for an arbitrary dimension is still unknown.

An alternative to this construction of boundary terms was proposed in [14] for odd dimensions and [15] for even dimensions, where the boundary terms have a geometrical origin (closely related to Chern-Simons forms), and that is based on boundary conditions that are not the standard Dirichlet one. For instance, even in $D=4$, a different boundary condition leads to a boundary term that regularizes the AdS action, but that does not recover the Gibbons-Hawking term plus Dirichlet counterterms, as a consequence of a different finite action principle. But, at the same time, this boundary term is dictated by the Euler theorem, showing the profound connection with topological invariants.

This paper understands the simplest example of the odd-dimensional regularization scheme proposed in [14]. Even though the explicit relation to the Dirichlet problem is possible here, a comparison in the general case is still unknown. The guideline to achieve finite conserved charges and Euclidean action is a well-defined action principle for a boundary condition on the extrinsic curvature. In spite of the simplicity of 3D, the suitability of this boundary condition for higher odd-dimensional gravity becomes evident from its compatibility with the Dirichlet problem in AdS gravity.

2 The action principle

We consider three-dimensional AdS gravity described by the action

$$I=-\frac{1}{16\pi G_N}\left[\int_M d^3x\,\sqrt{-G}\left(\hat{R}+\frac{2}{\ell^2}\right)+2\alpha\int_{\partial M}d^2x\,\sqrt{-h}K\right], \tag{1}$$

where ℓ is the AdS radius and we have supplemented the bulk Lagrangian by a boundary term that is α times the Gibbons-Hawking term [16].

As it is standard in holographic renormalization [9], we take a Gaussian (normal) form for the spacetime metric

$$ds^2=G_{\mu\nu}dx^\mu dx^\nu=N^2(\rho)\,d\rho^2+h_{ij}(\rho,x)dx^idx^j\,, \tag{2}$$

such that the only relevant boundary is at $\rho=const$. However, we shall not take any particular expansion for the boundary metric $h_{ij}(\rho,x)$.

We will work in the language of differential forms, with the dreibein $e^A=e^A_\mu\,dx^\mu$ (the spacetime metric is $G_{\mu\nu}=\eta_{AB}\,e^A_\mu e^B_\nu$) and the spin connection $\omega^{AB}=\omega^{AB}_\mu\,dx^\mu$ because certain features of the theory become manifest in terms of differential forms, as we shall see below.

In order to preserve the Lorentz covariance of the boundary term, we introduce the second fundamental form (SFF) as the difference between the dynamical field ω^{AB} and a fixed spin connection $\bar{\omega}^{AB}$,

$$\theta^{AB}=\omega^{AB}-\bar{\omega}^{AB}\,. \tag{3}$$

For the gauge (2), the dreibein adopts the block form $e^1 = N d\rho$ and $e^a = e^a_i\, dx^i$ with the indices splitting $A = \{1, a\}$. The spin connection decomposes as $\omega^{AB} = \{\omega^{1a}, \omega^{ab}\}$. For the torsionless case, the block ω^{ab} is related to the Christoffel symbol $\hat{\Gamma}^k_{ij}(G) = \Gamma^k_{ij}(h)$ of the boundary metric h_{ij}, that transforms as a connection (and not a tensor) in the boundary indices, so that it cannot enter the boundary term explicitly. On the other hand, for the rest of the components on ∂M, we have

$$\omega^{1a} = K^j_i\, e^a_j\, dx^i = K^a\,, \tag{4}$$

where the extrinsic curvature K_{ij} in normal coordinates (2) is given by

$$K_{ij} = N\, \hat{\Gamma}^\rho_{ij} = -\frac{1}{2N}\, \partial_\rho h_{ij}. \tag{5}$$

The explicit dependence on ω^{ab} can be removed by taking $\bar{\omega}^{AB}$ as coming from a product metric

$$ds^2 = \bar{N}^2(\rho)\, d\rho^2 + \bar{h}_{ij}(x)\, dx^i dx^j \tag{6}$$

cobordant to the dynamical one, i.e., it matches h_{ij} only on the boundary, $\bar{h}_{ij}(x) = h_{ij}(\rho_0, x)$ and such that this spin connection on ∂M contains only tangential components [17, 18],

$$\bar{\omega}^{1a} = 0\,, \qquad \bar{\omega}^{ab} = \omega^{ab}. \tag{7}$$

Thus, the SFF can be used to express all the quantities as boundary tensors (e.g., the extrinsic curvature),

$$\theta^{1a} = K^a_i\, dx^i, \qquad \theta^{ab} = 0\,. \tag{8}$$

The explicit form taken by the SFF in normal coordinates (2) is the key point to obtain the boundary term in the Euler theorem in four dimensions [17]. This argument has also been used to obtain the boundary term that regularizes AdS gravity in higher odd [14] and even [15] dimensions.

With the above definitions, the action (1) can be written

$$I = \frac{1}{16\pi G_N}\left[\int_M \varepsilon_{ABC}\left(\hat{R}^{AB} + \frac{1}{3\ell^2}\, e^A e^B\right) e^C - \alpha \int_{\partial M} \varepsilon_{ABC}\, \theta^{AB} e^C\right], \tag{9}$$

in terms of the Lorentz curvature $\hat{R}^{AB} = \frac{1}{2}\, \hat{R}^{AB}_{\mu\nu}\, dx^\mu \wedge dx^\nu = d\omega^{AB} + \omega^A_C \wedge \omega^{CB}$, the SFF, the triad and the Levi-Civita tensor, defined as $\varepsilon_{012} = -1$. We omit the wedge product between differential forms.

An arbitrary variation of this action, projected in the frame (2), produces the surface term

$$\delta I = -\frac{1}{8\pi G_N} \int_{\partial M} \varepsilon_{ab} \left[(1-\alpha)\delta K^a e^b - \alpha K^a \delta e^b\right], \tag{10}$$

when equations of motion hold. The Levi-Civita tensor in two dimensions is defined as $\varepsilon_{ab} = -\varepsilon_{1ab}$. We also used the fact that any variation acting on the SFF is $\delta\theta^{AB} = \delta\omega^{AB}$, as $\bar{\omega}^{AB}$ is kept fixed on the boundary ∂M.

In a radial foliation of the spacetime (2) the boundary metric and the extrinsic curvature are independent variables. In fact, K_{ij} is closely related to the conjugate momentum of h_{ij}, where the radial coordinate plays the role of time. Standard choice $\alpha = 1$ clearly recovers the Gibbons-Hawking term and defines the Dirichlet problem for gravity, because it eliminates the

variation of K^a and replaces it by a variation of the boundary dreiben e^b, producing the surface term

$$\delta I_D = \frac{1}{16\pi G_N}\int_{\partial M} d^2x\,\sqrt{-h}\left(K^{ij} - h^{ij}K\right)\delta h_{ij}\,. \tag{11}$$

This choice of α ensures a well-posed action principle for arbitrary variations of the boundary metric h_{ij}. However, the action I_D requires a counterterm

$$I_{reg} = I_D + \frac{1}{8\pi G_N}\int_{\partial M} d^2x\,\frac{1}{\ell}\,\sqrt{-h}\,, \tag{12}$$

to achieve the finiteness of both the Euclidean action and the conserved quantities [10] obtained through a quasilocal (boundary) stress tensor definition [19].

Here, we shall consider a different coefficient $\alpha = 1/2$ and analyze the consequences of this choice. As it can be seen from Eq.(10), the surface term takes the form

$$\delta I = -\frac{1}{16\pi G_N}\int_{\partial M} \varepsilon_{ab}\left(\delta K^a e^b - K^a \delta e^b\right) \tag{13}$$

that, with the help of $\delta K^a = \left(\delta K_i^j e_j^a + K_i^j \delta e_j^a\right) dx^i$, can be written as

$$\delta I = -\frac{1}{16\pi G_N}\int_{\partial M} d^2x\,\varepsilon_{ab}\varepsilon^{ik}\left[\delta K_i^j e_j^a e_k^b + \delta e_j^a e_l^b\left(K_i^j\delta_k^l - K_k^l\delta_i^j\right)\right]. \tag{14}$$

In this case, the action becomes stationary only under a suitable boundary condition on the extrinsic curvature K_i^j.

3 Asymptotic conditions

We consider fixing the extrinsic curvature on the boundary ∂M, that is,

$$\delta K_i^j = 0\,, \tag{15}$$

in order to cancel the first term in Eq.(14). This means that, in the asymptotic region, K_i^j tends to a (1,1)-tensor with vanishing variation. For simplicity, we take

$$K_i^j = \frac{1}{\ell}\,\delta_i^j\,, \tag{16}$$

where the $1/\ell$ factor is introduced in order to fix the scale for asymptotically AdS (AAdS) spacetimes. This choice makes the rest of the surface term in Eq.(14) vanish identically, so that the gravitational action has indeed an extremum for that boundary condition.

To further understand the meaning of the condition (16), we can put Eq.(5) in the form

$$K_{ij} = -\frac{1}{2}\,n^\mu\partial_\mu h_{ij} = -\frac{1}{2}\,\mathcal{L}_n h_{ij}\,, \tag{17}$$

where $\mathcal{L}_n$ is a directional (Lie) derivative along a unit vector normal to the boundary, $n_\mu = (0, N, 0)$. Inserting the definition (17) in the asymptotic condition (16), we see that the latter relation is satisfied in a spacetime whose boundary ∂M is endowed with a conformal Killing vector because

$$\mathcal{L}_n h_{ij} = \hat{\nabla}_i n_j + \hat{\nabla}_j n_i = -\frac{2}{\ell}\,h_{ij}\,. \tag{18}$$

A submanifold whose extrinsic curvature is proportional to the induced metric is usually referred to as *totally umbilical* [18].

In order to describe AAdS spacetimes, it is common to take the lapse function as $N=\ell/2\rho$ and the boundary metric as $h_{ij}(\rho,x)=g_{ij}(\rho,x)/\rho$, so that

$$ds^2=\frac{\ell^2}{4\rho^2}\,d\rho^2+\frac{1}{\rho}\,g_{ij}\left(\rho,x\right)dx^idx^j\,, \tag{19}$$

that is suitable to represent the conformal structure of the boundary located at $\rho=0$. According to Fefferman and Graham [20], the metric $g_{ij}\left(\rho,x\right)$ is regular on the boundary and it can be expanded around $\rho=0$ as

$$g_{ij}\left(\rho,x\right)=g_{(0)ij}\left(x\right)+\rho g_{(1)ij}\left(x\right)+\rho^2g_{(2)ij}\left(x\right)+\cdots\,, \tag{20}$$

where $g_{(0)ij}$ is a given initial data for the metric. In three dimensions, the Weyl tensor vanishes identically and the FG series (20) becomes finite, terminating at order ρ^2 [21]. The solution of the Einstein equation in this case is $g_{(2)ij}=\frac{1}{4}\,(g_{(1)}g_{(0)}^{-1}g_{(1)})_{ij}$, where $g_{(1)ij}$ has the trace fixed in terms of the curvature of $g_{(0)ij}$.

The standard Dirichlet boundary condition on h_{ij} is in general ill-defined for AdS gravity because of its conformal boundary. Indeed, it follows from its asymptotic form (19,20) that the induced metric is divergent at the boundary and therefore, it is not suitable to fix it there. Alternatively, one can demand that a conformal structure (i.e., its representative $g_{(0)ij}$) is kept fixed at the boundary. As discussed in [12], this action principle requires the addition of new boundary terms apart from the usual Gibbons-Hawking term. However, it can be proven that these extra terms are indeed the usual Dirichlet counterterm series.

The compatibility of the boundary condition (16) with the Fefferman-Graham form of the metric (19,20) is then evident from the expansion of the extrinsic curvature (5),

$$K_i^j=\frac{1}{\ell}\,\delta_i^j-\frac{\rho}{\ell}\,g_{(1)ik}\,g_{(0)}^{kj}+\cdots\,, \tag{21}$$

that contains only increasing powers of ρ. This also implies that fixing the extrinsic curvature at the boundary is equivalent to keeping fixed the conformal structure. As a consequence, the Dirichlet problem for the conformal metric as the boundary data can be converted into the initial-value problem for K_{ij}, such that the standard holographic renormalization can be reformulated in terms of the extrinsic curvature [22].

As we shall see below, in the present case the regularization is encoded in the boundary term that extremizes the action for the boundary condition (16).

4 Regularized action

For $\alpha=1/2$, the action is written as

$$I=-\frac{1}{16\pi G_N}\left[\int_M d^3x\,\sqrt{-G}\left(\hat{R}+\frac{2}{\ell^2}\right)+\int_{\partial M}d^2x\,\sqrt{-h}K\right]. \tag{22}$$

Its variation on-shell is the surface term (13), containing both variations of the boundary dreibein e^a and the extrinsic curvature K^a. The formulation in terms of these variables is useful to recover the conserved quantities displayed below from a generic Chern-Simons theory in three dimensions.

The Noether current can be written as [23, 24]

$$*J = -\Theta(e^a, K^a, \delta e^a, \delta K^a) - i_\xi \left(L + dB\right) , \tag{23}$$

where Θ is the surface term in the variation of the action (13), L and B are the bulk Lagrangian and the boundary term in Eq.(22), respectively, and i_ξ is the contraction operator with the Killing vector ξ^μ [25]. The Noether charges, with the contributions coming from the bulk and the boundary, are then given by

$$\begin{aligned} Q(\xi) &= \mathcal{K}(\xi) + \int_{\partial\Sigma} \left(i_\xi K^a \frac{\delta B}{\delta K^a} + i_\xi e^a \frac{\delta B}{\delta e^a} \right) \\ &= \mathcal{K}(\xi) - \frac{1}{16\pi G_N} \int_{\partial\Sigma} \varepsilon_{ab} \left(i_\xi K^a e^b - K^a i_\xi e^b \right) . \end{aligned} \tag{24}$$

The first term is known as the Komar's integral

$$\mathcal{K}(\xi) = \frac{1}{8\pi G_N} \int_{\partial\Sigma} \varepsilon_{ab}\, i_\xi K^a e^b , \tag{25}$$

and it is the conserved quantity associated to the bulk term in the gravity action.

Finally, the conserved quantities for three-dimensional AdS gravity read

$$\begin{aligned} Q(\xi) &= \frac{1}{16\pi G_N} \int_{\partial\Sigma} \varepsilon_{ab} \left(i_\xi K^a e^b + K^a i_\xi e^b \right) \\ &= \frac{1}{16\pi G_N} \int_{\partial\Sigma} \sqrt{-h}\, \varepsilon_{ij}\, \xi^k \left(\delta^j_l K^i_k + \delta^j_k K^i_l \right) dx^l . \end{aligned} \tag{26}$$

Stationary, circularly symmetric black holes exist in three-dimensional gravity only in presence of negative cosmological constant. The metric for the BTZ black hole [26] reads

$$ds^2 = -\gamma(r) f^2(r)\, dt^2 + \frac{dr^2}{f^2(r)} + r^2 \left(d\varphi + n(r) dt \right)^2 , \tag{27}$$

with

$$f^2(r) = -8G_N\, M + \frac{r^2}{\ell^2} + \frac{16 G_N^2 J^2}{r^2} , \qquad n(r) = -\frac{4G_N J}{r^2} \qquad \gamma(r) = 1. \tag{28}$$

The horizon r_+ is defined by the largest radius satisfying $f(r_+) = 0$.

For the isometries ∂/∂_t and $\partial/\partial_\varphi$, the charge formula (26) provides the correct conserved quantities for the BTZ metric,

$$Q(\partial_t) = M , \qquad Q(\partial_\varphi) = J , \tag{29}$$

where $\partial\Sigma$ is taken as S^1 at radial infinity. The vacuum energy for three-dimensional AdS space corresponds to $M = -1/8G_N$. On the contrary to the Hamiltonian approach [26] or perturbative Lagrangian methods [27], we do not need to specify the background to obtain the correct results (29).

The *regularized* action (22) does not lend itself for a clear definition of a boundary stress tensor T^{ij} because its variation (13) contains a piece along δK^a that it is usually cancelled by the Gibbons-Hawking term. However, we can rewrite the action as

$$I = I_D + \frac{1}{16\pi G_N} \int_{\partial M} d^2x \sqrt{-h}\, K , \tag{30}$$

where I_D stands for the action suitable for the Dirichlet problem (Eq.(1) with $\alpha = 1$). We will consider now the extra term in (30) as a functional of the boundary metric $h_{ij}(\rho, x) = g_{ij}(\rho, x)/\rho$. The extrinsic curvature (5) can be generically written as

$$K_i^j = \frac{1}{\ell}\,\delta_i^j - \frac{\rho}{\ell}\,k_i^j\,, \tag{31}$$

with $k_i^j = g^{jk}\partial_\rho g_{ki}$, so that the second term in Eq.(30) takes the form

$$\frac{1}{16\pi G_N}\sqrt{-h}K = \frac{1}{8\pi G_N \ell}\left(\sqrt{-h} - 2\sqrt{-g}\,k\right)\,. \tag{32}$$

The first term is just the Balasubramanian-Kraus counterterm [10], whereas the second one can be shown to be a topological invariant of the boundary metric $g_{(0)}$, that is, $\sqrt{-g_{(0)}}\,R_{(0)}$. This follows from the fact that $-2\sqrt{-g}\,k = -2\sqrt{-g_{(0)}}\,\mathrm{Tr}(g_{(1)})$ on the boundary. Indeed, the 3D Einstein equation in the gauge (19) determines the trace and vanishing covariant divergence of $g_{(1)ij}$ [9]. Then, the boundary term in Eq.(22) both regularizes the quasilocal stress tensor and reproduces the correct Weyl anomaly [38].

The above argument also explains why the Euclidean action supplemented by a Gibbons-Hawking term with an *anomalous* factor is finite, as first noticed in [28] where it correctly describes the thermodynamics of the BTZ black hole. It has been shown in [12] that the counterterms constructed in the regularization procedure in terms of the extrinsic curvature [22] (and that are equivalent to standard counterterms) allows to prove the first law of black hole thermodynamics for a general asymptotically AdS black hole. Here, it follows from the equivalence of the boundary term in (22) to the Dirichlet counterterms plus a topological invariant that the right thermodynamics is recovered in a general case.

5 Chern-Simons formulation

The boundary term in (22) arises naturally in the Chern-Simons formulation of three-dimensional AdS gravity [28]. Indeed, the Chern-Simons (CS) action

$$I_{CS}\,[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(AdA + \frac{2}{3}\,A^3\right) \tag{33}$$

for the AdS group $SO(2,2)$ whose gauge connection is given by

$$A_{AdS} = \frac{1}{2}\,\omega^{AB}J_{AB} + \frac{1}{\ell}\,e^A P_A\,, \tag{34}$$

and the trace of the AdS generators set $\mathrm{Tr}(J_{AB}P_A) = \varepsilon_{ABC}$, is equivalent to the Einstein-Hilbert-AdS bulk action plus the Bañados-Mendez boundary term

$$\frac{1}{16\pi G_N}\int_{\partial M}\omega_A\,e^A = \frac{1}{16\pi G_N}\int_{\partial M} d^2x\,\sqrt{-h}\,K \tag{35}$$

in the coordinate frame (2) (here $\omega_A = \frac{1}{2}\,\varepsilon_{ABC}\,\omega^{BC}$). Clearly, the boundary term (35) is not Lorentz-covariant as the one constructed up with the SFF in Eq.(9). This is an accident that happens only in $(2+1)$ dimensions: the dreibein cannot go along $d\rho$ at the boundary and so the boundary term does not depend on ω^{ab}. Therefore, the residual $2D$ Lorentz symmetry on ∂M permits to express (35) as tensors on the boundary (the metric h_{ij} and the extrinsic

curvature K_{ij}). In an arbitrary local Lorentz frame, the non-invariance of the boundary term under local Lorentz transformations produces extra asymptotic degrees of freedom responsible for the arbitrary coupling constant λ in the potential term of Liouville theory [29], that is either zero or put by hand in a metric formulation.

In higher odd dimensions, one can also pass from the CS formulation for the AdS group $SO(2n,2)$ in terms of the connection A to a Lovelock-type Lagrangian for gravity, i.e., a polynomial in the Riemann two-form and the metric [30]. In doing so, however, the produced boundary term will be neither Lorentz-covariant nor the correct one that regulates the conserved quantities and the Euclidean action for CS black holes [31]. The introduction of the SFF is then essential to restore Lorentz covariance and it also provides a clear guideline for its explicit construction [32].

Usually, the Chern-Simons formulation for $SO(2,2)$ exploits the fact that the AdS gravity action can be written as the difference of two copies of the CS action (33) for $SO(2,1)$ [5] (in the Euclidean case, $SL(2,\mathbb{C})$)

$$I = I_{CS}\left[A\right] - I_{CS}\left[\bar{A}\right] \,, \tag{36}$$

where the connections for each copy of $SO(2,1)$ are

$$A^A = \omega^A + \frac{1}{\ell}\,e^A, \qquad \bar{A}^A = \omega^A - \frac{1}{\ell}\,e^A, \tag{37}$$

with $k = -\ell/4G_N$.

The variation of the action (36) produces the equations of motion plus a surface term that is cancelled by taking *chiral* boundary conditions

$$A_{\bar{z}} = 0 \quad \text{and} \quad \bar{A}_z = 0\,, \tag{38}$$

with the use of the light-cone coordinates $z = t + \ell\varphi$ and $\bar{z} = t - \ell\varphi$ for Minkowskian signature [2] (the Euclidean version considers the same set of boundary conditions, but for complex coordinates $(z,\bar{z})$ defined on the solid torus that describes the topology of the Euclidean black hole [28, 33]). In the Chern-Simons formulation, the explicit form of the conditions (38) is

$$2\ell A^A_{\bar{z}} \;=\; \left(\ell\,\omega^A_t - \frac{1}{\ell}\,e^A_\varphi\right) - \left(\omega^A_\varphi - e^A_t\right) = 0\,, \tag{39}$$

$$2\ell \bar{A}^A_z \;=\; \left(\ell\,\omega^A_t - \frac{1}{\ell}\,e^A_\varphi\right) + \left(\omega^A_\varphi - e^A_t\right) = 0\,, \tag{40}$$

that are satisfied by AdS gravity. Indeed, the three-dimensional black hole has

$$\begin{aligned} e^0 &= f dt\,, & e^1 &= \frac{1}{f}\,dr\,, & e^2 &= rN^\varphi dt + rd\varphi\,, \\ \omega^0 &= f d\varphi\,, & \omega^1 &= \frac{J}{2r^2 f}\,dr\,, & \omega^2 &= rN^\varphi d\varphi + \frac{r}{\ell^2}\,dt\,. \end{aligned} \tag{41}$$

Thus, the action has an extremum for the chiral boundary conditions, eqs.(39,40), that is clearly not the standard Dirichlet one for the metric. This shows that there are (at least) two ways of regularizing the AdS gravity in three dimensions. In this paper, we consider another boundary condition that also explains the anomalous factor in the Gibbons-hawking term.

On the contrary to the relations (39,40) fulfilled by the bulk geometry, the boundary condition (16) implies

$$\omega^{a1}_i = \frac{1}{\ell}\,e^a_i \tag{42}$$

only in the asymptotic region ($a=\{0,2\}$ and $i=\{t,\varphi\}$). In fact, in the CS formulation of 3D AdS gravity, the surface term coming from an arbitrary variation is

$$\begin{aligned}\delta I_{CS}\left[A_{AdS}\right] &= -\frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(A\delta A\right) && (43)\\ &= \frac{1}{32\pi G_N}\int_{\partial M}\varepsilon_{ABC}\left(\delta\omega^{AB}e^{C}-\omega^{AB}\delta e^{C}\right)\,, && (44)\end{aligned}$$

that reduces to Eqs.(13,14) for the radial foliation (2).

The AdS connection can also be written as $A=\frac{1}{2}W^{\bar{A}\bar{B}}J_{\bar{A}\bar{B}}$ using covering space indices $\bar{A}=\{A,3\}$, where $W^{AB}=\omega^{AB}$ and $W^{A3}=\frac{1}{\ell}e^{A}$. Then, the asymptotic condition (42) adopts the compact form

$$W_i^{a1}=W_i^{a3}. \tag{45}$$

This new condition might have nontrivial consequences at the level of the induced theory at the boundary.

The surface term (44) was obtained in [34] for the Palatini form of the AdS gravity action

$$\delta I_g=\frac{1}{32\pi G_N}\int_{\partial M}n_\mu\left[\left(\hat{\Gamma}^{\lambda}_{\nu\lambda}\delta\mathcal{G}^{\mu\nu}-\hat{\Gamma}^{\mu}_{\nu\lambda}\delta\mathcal{G}^{\nu\lambda}\right)-\left(\mathcal{G}^{\mu\nu}\delta\hat{\Gamma}^{\lambda}_{\nu\lambda}-\mathcal{G}^{\nu\lambda}\delta\hat{\Gamma}^{\mu}_{\nu\lambda}\right)\right]\,, \tag{46}$$

where $\mathcal{G}^{\mu\nu}=\sqrt{-G}G^{\mu\nu}$. Clearly, the action has an extremum for a mixed Dirichlet-Neumann boundary condition. However, no clear identification of such boundary condition in terms of tensorial quantities defined on ∂M was made in this reference.

In CS formulation it is simple to obtain the Noether charges. With the surface term (43) in terms of the gauge connection, we can compute the conserved charges associated to an asymptotic Killing vector ξ using the Noether theorem. The conserved current for this theory is

$$*J=\frac{k}{4\pi}\left[-\mathrm{Tr}\left(A\mathcal{L}_\xi A\right)-i_\xi\mathrm{Tr}\left(AF-\frac{1}{3}A^3\right)\right]\,. \tag{47}$$

The Lie derivative for a connection field takes the form $\mathcal{L}_\xi A=D(i_\xi A)+i_\xi F$, where the covariant derivative is defined as $D(i_\xi A)=d(i_\xi A)+[A,i_\xi A]$. Using the equation of motion $F=dA+A^2=0$, and integrating by parts, the current is finally expressed as

$$*J=\frac{k}{4\pi}\,d\mathrm{Tr}\left(AI_\xi A\right)\,, \tag{48}$$

from where we can read the conserved charge as [35, 36]

$$Q(\xi)=\frac{k}{4\pi}\int\limits_{\partial\Sigma}\mathrm{Tr}\left(Ai_\xi A\right)\,. \tag{49}$$

It is not difficult to prove that this expression recovers the formula (26) for the trace of AdS generators and the radial foliation considered above.

6 Conclusions

In this paper, we show how a single boundary term solves at once three problems in three-dimensional AdS gravity: it defines a well-posed variation of the action (the action has extremum

on-shell under the condition (16)), produces finite charges and regularizes the Euclidean action. In other words, the Dirichlet counterterm is built-in in a boundary term that is 1/2 of the Gibbons-Hawking term.

A boundary condition on the extrinsic curvature (16), equivalent to keeping fixed the metric of the conformal boundary in AAdS spacetimes, ensures a finite action principle in agreement to the Dirichlet counterterms problem.

The same boundary condition leads to a regularization scheme of AdS gravity alternative to the standard counterterms procedure [9, 10, 11, 39, 40]. In the new approach, the boundary terms can depend, apart from intrinsic quantities constructed out of the boundary metric h_{ij} and boundary curvature R_{ij}^{kl}, also on the extrinsic curvature K_{ij}. At first, one might think that the number of possible counterterms constructed up with these tensors is even higher than in the standard procedure. However, the boundary condition (16) and another one on the asymptotic curvature –that is identically satisfied in FG frame– are restrictive enough to substantially reduces the counterterms series to a compact expression [14, 15]. For instance, in five dimensions the boundary term that regularizes the AdS action is

$$I=-\frac{1}{16\pi G_N}\int_M d^5x\,\sqrt{-G}\,(\hat{R}-2\Lambda)+c_4\int_{\partial M}B_4\,, \tag{50}$$

with $c_4 = const.$ and the boundary term given by the expression

$$B_4=-\frac{1}{2}\,\sqrt{-h}\,\delta^{[i_1i_2i_3i_4]}_{[j_1j_2j_3j_4]}K^{j_1}_{i_1}\delta^{j_2}_{i_2}\left(R^{j_3j_4}_{i_3i_4}(h)-K^{j_3}_{i_3}K^{j_4}_{i_4}+\frac{1}{3\ell^2}\,\delta^{j_3}_{i_3}\delta^{j_4}_{i_4}\right). \tag{51}$$

It is worthwhile noticing that B_4 contains a term proportional to $\sqrt{-h}\,K$, with a numerical factor that again differs from the one of the Gibbons-Hawking term. The variation of the above action –on-shell– takes the form

$$\begin{aligned}\delta I &= 2\int_{\partial M}\varepsilon_{abcd}\,\delta K^a e^b\left[\kappa\,e^ce^d+c_4\left(\hat{R}^{cd}+\frac{1}{3\ell^2}\,e^ce^d\right)\right]\\ &\quad-\frac{c_4}{2}\,\varepsilon_{abcd}\left(\delta K^a e^b-K^a\delta e^b\right)\left(R^{cd}-\frac{1}{2}K^cK^d+\frac{1}{2\ell^2}\,e^ce^d\right)\,,\end{aligned} \tag{52}$$

where $\kappa = 1/(96\pi G_N)$ and $\hat{R}^{cd} = R^{cd} - K^cK^d$ is the Gauss-Coddazzi relation for the Riemann tensor. An appropriate choice of the coupling constant, $c_4 = 3\kappa\ell^2/2$, makes the first line of above equation proportional to the AdS curvature $\hat{R}^{cd} + \frac{1}{\ell^2}\,e^ce^d$, that vanishes at the boundary for AAdS spacetimes.[1] Then, the second line is proportional to

$$\varepsilon_{abcd}\,\varepsilon^{i_1i_2i_3i_4}\left[\delta K^j_{i_1}e^a_je^b_{i_2}+\delta e^a_je^b_l\left(K^j_{i_1}\delta^l_{i_2}-K^l_{i_2}\delta^j_{i_1}\right)\right]\left(R^{cd}_{i_3i_4}-K^c_{i_3}K^d_{i_4}+\frac{1}{\ell^2}\,e^c_{i_3}e^d_{i_4}\right)\,, \tag{53}$$

that is again cancelled by taking the boundary condition (15,16). This shows that, on the contrary to standard conditions (39,40) for $3D$ AdS gravity, the condition (16) can indeed be lifted to higher odd-dimensional AdS gravity. The boundary term derived from this action principle equally cancels the infinities in the Euclidean action and conserved quantities [14].

The action principle for three-dimensional AdS gravity presented here agrees with the Dirichlet problem up to a topological invariant at the boundary. Even though this observation is almost

[1] This condition on the asymptotic Riemann was considered for first time in [37] in order to have a finite action principle in even dimensions. It can be shown, however, that this asymptotic behavior is also implied by the Fefferman-Graham expansion for that tensor [15].

trivial in three dimensions, we could expect that the boundary terms in ref.[14] (for $D = 2n+1$) and ref.[15] (for $D = 2n$) generate the full series of standard counterterms carrying out a suitable expansion.

Acknowledgments

We wish to thank M. Bañados and S. Theisen for helpful discussions. This work was partially funded by the grants 3030029 and 3040026 from FONDECYT (Chile) and by Fundação para a Ciência e Tecnologia (FCT) of the Ministry of Science, Portugal, through project POCTI/FNU/44648/2002. O.M. is also supported by PUCV through the program Investigador Joven 2006.

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